Journal of Engineering Physics and Thermophysics, Vol. 72, No. 5, 1999

SOLUTION OF A PROBLEM CONCERNING THE DISTRIBUTION OF GAS MOLECULES IN A LAYER WITH MIRROR BOUNDARY CONDITIONS

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In the present work we investigate the distribution of gas molecules in a layer filled with a rarefied gas. To solve the problem, a Boltzmann model nonstationary equation is used. The distribution function is found in the form of an expansion in generalized eigenfunctions of the corresponding characteristic equation.

Boundary-value problems for kinetic equations arise in the solution of physical problems in such fields of science as the kinetic theory of gas and plasma, the theory of neutron transfer, in the experimental study of ultrasonic wave dispersion in a layer, etc.

A review of attempts undertaken to solve analytically kinetic equations for problems of a critical layer in the theory of nuclear reactors, Couette and Poiseuille problems, and for other problems was given in [1-4]. We note that in these investigations the authors used numerical-analytical methods.

In the present work an exact solution of the boundary-value problem is obtained for a kinetic equation in a layer in the case when both plates bounding the gas perform harmonic oscillations (the lower plate, with an arbitrary constant amplitude and frequency, and the upper plate, with the forced ones). Here the well-known Case–Zweifel method [1] is modified. It should be noted that the modification of this method made it possible to solve a variety of problems for model kinetic equations that for a long time have not been accurately calculable. Among these are the problem of calculation of a temperature jump [5], the Landau problem on the behavior of an electron plasma in a layer [6] (this problem was exactly solved by Landau for a half-space), and the problem of strong evaporation for one-dimensional [7] and three-dimensional [8] gases.

We note that in [9, 10] an attempt was made to develop the Case–Zweifel method for exact solution of a half-space boundary-value problem. However, a theory constructed on the basis of a procedure of Abelian differentials on Riemann surfaces is so complex that up to now it has not been used in solving applied problems. For comparison we point out that the method developed in the present work allows one to construct in closed form a function for the velocity distribution of molecules for a rarefied gas in a layer.

1. Statement of the Problem. We consider a layer of thickness d filled with a rarefied gas. The lower plate bounding the gas lies in the plane $x = 0$, and the upper plate, in the plane $x = d$. The x axis is perpendicular to the plates. The lower plate performs normal harmonic oscillations with frequency ω and amplitude U ($x = U \exp (i[\omega t])$) relative to its equilibrium position $x = 0$. The upper plate performs forced harmonic oscillations with frequency ω, amplitude U_d, and initial phase φ_0 ($x = U_d \exp (i[\omega t + \varphi_0])$). It is required to construct the gas-molecule distribution function.

Let us take a Boltzmann model kinetic equation with a collision operator in the form suggested by Bhatnagar, Gross, and Crook (see, for example, [11]):

$$\left(\frac{\partial}{\partial t} + \mu \frac{\partial}{\partial x} + 1\right) Y(t, x, \mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\mu'^2)\, Y(t, x, \mu')\, d\mu' \qquad (1)$$

(μ is the projection of the molecular velocity onto the x axis). The boundary conditions are obtained from the following condition of the problem:

Moscow Pedagogical University, Russia; Vologda Science Coordination Center of the Russian Academy of Sciences, Moscow, Russia. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 72, No. 5, pp. 862-869, September-October, 1999. Original article submitted June 30, 1998; revision submitted November 30, 1998.

$$Y(t, 0, \mu) = Y(t, 0, -\mu) + 2U\mu \exp(i\omega t), \quad t > 0, \quad \mu > 0;$$

$$Y(t, d, \mu) = Y(t, d, -\mu) + 2U_d \mu \exp(i[\omega t + \varphi_0]), \quad t > 0, \quad \mu < 0.$$

Considering the process to be stationary, we separate the time variable, assuming that

$$Y(t, x, \mu) = \Psi(x, \mu) \exp(i\omega t). \tag{2}$$

Having substituted Eq. (2) into Eq. (1), we reduce the nonstationary boundary-value problem to a stationary one

$$\left(i\omega + \mu \frac{\partial}{\partial x} + 1\right) \Psi(x, \mu) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\mu'^2) \Psi(x, \mu') \, d\mu' . \tag{3}$$

The boundary conditions are rearranged to the form:

$$\Psi(0, \mu) = \Psi(0, -\mu) + 2U\mu, \quad \mu > 0;$$
$$\Psi(d, \mu) = \Psi(d, -\mu) + 2U_d \mu \exp(i\omega_0), \quad \mu < 0. \tag{4}$$

We will next consider a boundary-value problem that consists in solving Eq. (3) with boundary conditions (4).

2. Characteristic System of Equations. Eigenfunctions. To derive the characteristic system of equations, we will use the procedure described in [12].

We separate the variables in Eq. (3) in the following manner:

$$\Psi_\eta(x, \mu) = \exp\left[-\frac{x}{\eta}(1 + i\omega)\right] \Phi_1(\eta, \mu) + \exp\left[-\frac{d - x}{\eta}(1 + i\omega)\right] \Phi_2(\eta, \mu). \tag{5}$$

Here $\eta \in \mathbf{C}$ (**C** is the complex plane), $\mu > 0$. Substituting Eq. (5) into (3) and taking the following normalization conditions:

$$(1 + i\omega) n_k(\eta) = \int_{-\infty}^{\infty} \exp(-\mu^2) \Phi_k(\eta, \mu) \, d\mu \quad (k = 1, 2), \tag{6}$$

we obtain a characteristic system of equations:

$$(\eta - \mu) \Phi_1(\eta, \mu) = \frac{1}{\sqrt{\pi}} \eta n_1(\eta), \quad (\eta + \mu) \Phi_2(\eta, \mu) = \frac{1}{\sqrt{\pi}} \eta n_2(\eta), \tag{7}$$

the solution of which depends substantially on whether the spectral parameter η belongs to the real axis or not. We consider two cases.

1. Let $\eta \notin \mathbf{R}$. In this case the eigenfunctions have the form:

$$\Phi_1(\eta, \mu) = \frac{1}{\sqrt{\pi}} \eta \frac{1}{(\eta - \mu)} n_1(\eta), \quad \Phi_2(\eta, \mu) = \frac{1}{\sqrt{\pi}} \eta \frac{1}{(\eta + \mu)} n_2(\eta). \tag{8}$$

Substituting Eq. (8) into Eq. (6), we obtain conditions superimposed on the eigenfunctions of discrete spectrum (8): $\Lambda(z; \omega) = 0$, where

$$\Lambda(z; \omega) = \lambda_{\text{eig.f}}(z) + i\omega, \quad \lambda_{\text{eig.f}}(z) = 1 + z \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\mu^2) \frac{d\mu}{\mu - z}$$

The dispersion function $\Lambda(z; \omega)$, its zeros, and its properties were investigated in [13].

2. Let $\eta \in \mathbf{R}$. We find a solution of system (7) in a class of generalized functions [14]:

$$\Phi_1(\eta,\mu) = \frac{1}{\sqrt{\pi}}\eta P\frac{1}{\eta-\mu} n_1(\eta) + g_1(\eta)\,\delta(\eta-\mu)\,,$$
$$\Phi_2(\eta,\mu) = \frac{1}{\sqrt{\pi}}\eta P\frac{1}{\eta+\mu} n_2(\eta) + g_1(\eta)\,\delta(\eta+\mu)\,. \qquad (9)$$

Here Px^{-1} denotes the distribution, i.e., the principal value of the Cauchy integral; $\delta(x)$ is the Dirac delta-function.

Substitution of Eq. (9) into normalization condition (6) allows us to find $g_{1,2}(\eta)$. Hence, system (9) is rearranged to the form:

$$\Phi_1(\eta,\mu) = \left[\frac{1}{\sqrt{\pi}}\eta P\frac{1}{\eta-\mu} + \exp(\eta^2)\,\Lambda(\eta;\omega)\,\delta(\eta-\mu)\right] n_1(\eta)\,,$$
$$\Phi_2(\eta,\mu) = \left[\frac{1}{\sqrt{\pi}}\eta P\frac{1}{\eta+\mu} + \exp(\eta^2)\,\Lambda(\eta;\omega)\,\delta(\eta+\mu)\right] n_2(\eta)\,. \qquad (10)$$

Suppose that

$$\Phi(\eta,\mu) = \frac{1}{\sqrt{\pi}}\eta P\frac{1}{\eta-\mu} + \exp(\eta^2)\,\Lambda(\eta;\omega)\,\delta(\eta-\mu)\,. \qquad (11)$$

Using equality (11), we rewrite Eq. (10) in the following form:

$$\Phi_1(\eta,\mu) = \Phi(\eta,\mu)\,n_1(\eta)\,,\quad \Phi_2(\eta,\mu) = \Phi(\eta,-\mu)\,n_2(\eta)\,. \qquad (12)$$

Thus, we obtain the eigenfunctions of discrete (8) and continuous (12) spectra.

3. Expansion of the Boundary-Value Problem in Eigenvectors. We will seek a solution of problem (3) and (4) in the form of an expansion in eigenfunctions of characteristic system (7)

$$\Psi(x,\mu,\omega) = a_1(n_0;\omega)\,\Phi_1(\eta_0,\omega)\exp\left[-\frac{x}{\eta_0}(1+i\omega)\right] +$$
$$+ a_2(\eta_0;\omega)\,\Phi_2(\eta_0,\omega)\exp\left[-\frac{d-x}{\eta_0}(1+i\omega)\right] +$$
$$+ \int_0^\infty A_1(\eta;\omega)\exp\left[-\frac{x}{\eta}(1+i\omega)\right]\Phi_1(\eta,\mu)\,d\eta +$$
$$+ \int_0^\infty A_2(\eta;\omega)\exp\left[-\frac{d-x}{\eta}(1+i\omega)\right]\Phi_2(\eta,\mu)\,d\eta\,, \qquad (13)$$

where $\mathrm{Re}(\frac{1+i\omega}{\eta_0}) > 0$.

Substituting the values of $x = 0$, $x = d$ into Eq. (13) and taking into account boundary conditions (4) and relation (11), we obtain:

$$2U\mu = \frac{1}{\sqrt{\pi}}\frac{2\eta_0\mu}{\eta_0^2-\mu^2}\left\{a_1(\eta_0;\omega)\,n_1(\eta_0) - a_2(\eta_0;\omega)\exp\left[-\frac{d}{\eta_0}(1+i\omega)\right]n_2(\eta_0)\right\} +$$
$$+ \int_0^\infty A_2(\eta;\omega)\exp\left[-\frac{d}{\eta}(1+i\omega)\right]n_2(\eta)\,[\Phi(\eta,-\mu) - \Phi(\eta,\mu)]\,d\eta +$$

$$+ \int_0^\infty A_1(\eta;\omega)\, n_1(\eta)\, [\Phi(\eta,\mu) - \Phi(\eta,-\mu)]\, d\eta\,, \tag{14}$$

$$2U_d \mu \exp(i\varphi_0) = \frac{1}{\sqrt{\pi}} \frac{2\eta_0 \mu}{\eta_0^2 - \mu^2} \left\{ \exp\left[-\frac{d}{\eta_0}(1 + i\omega) \right] a_1(\eta_0;\omega)\, n_1(\eta_0) - a_2(\eta_0;\omega)\, n_2(\eta_0) \right\} +$$

$$+ \int_0^\infty \exp\left[-\frac{d}{\eta}(1 + i\omega) \right] A_1(\eta;\omega)\, n_2(\eta)\, [\Phi(\eta,\mu) - \Phi(\eta,-\mu)]\, d\eta +$$

$$+ \int_0^\infty A_2(\eta;\omega)\, n_2(\eta)\, [\Phi(\eta,-\mu) - \Phi(\eta,\mu)]\, d\eta\,. \tag{15}$$

Now we introduce the one-sided functions:

$$\overset{+}{n}_k = \begin{cases} n_k(\eta)\,, & \eta \geq 0\,, \\ 0\,, & \eta < 0\,, \end{cases} \quad \overset{+}{A}_k(\eta;\omega) = \begin{cases} A_k(\eta;\omega)\,, & \eta \geq 0\,, \\ 0\,, & \eta < 0\,, \end{cases} \quad k = 1, 2\,.$$

Using expressions (14), (15) and the fact that $\Phi(\eta, \mu) = \Phi(-\eta, -\mu)$, we obtain a system of equations:

$$\begin{aligned} &\frac{1}{\sqrt{\pi}} \varphi(\mu;\omega) + \int_{-\infty}^{\infty} \Phi(\eta,\mu)\, n(\eta;\omega)\, d\eta = 2U\mu\,, \\ &\frac{1}{\sqrt{\pi}} \psi(\mu;\omega) + \int_{-\infty}^{\infty} \Phi(\eta,\mu)\, m(\eta;\omega)\, d\eta = 2U_d \mu \exp(i\varphi_0)\,, \end{aligned} \tag{16}$$

where

$$\varphi(\mu;\omega) = \frac{2\eta_0 \mu}{\eta_0^2 - \mu^2} \left[a_1(\eta_0;\omega)\, n_1(\eta_0) - a_2(\eta_0;\omega) \exp\left[-\frac{d}{\eta_0}(1 + i\omega) \right] n_2(\eta_0) \right];$$

$$\psi(\mu;\omega) = \frac{2\eta_0 \mu}{\eta_0^2 - \mu^2} \left[\exp\left[-\frac{d}{\eta_0}(1 + i\omega) \right] a_1(\eta_0;\omega)\, n_1(\eta_0) - a_2(\eta_0;\omega)\, n_2(\eta_0) \right];$$

$$\begin{aligned} n(\eta;\omega) = {}& \overset{+}{A}_1(\eta;\omega)\, \overset{+}{n}_1(\eta) - \overset{+}{A}_1(\omega;-\eta)\, \overset{+}{n}_1(-\eta) + \exp\left[\frac{d}{\eta}(1 + i\omega) \right] \overset{+}{A}_2(\omega;-\eta)\, \overset{+}{n}_2(-\eta) - \\ & - \exp\left[-\frac{d}{\eta}(1 + i\omega) \right] \overset{+}{A}_2(\eta;\omega)\, \overset{+}{n}_2(\eta)\,; \end{aligned} \tag{17}$$

$$\begin{aligned} m(\eta;\omega) = {}& \exp\left[-\frac{d}{\eta}(1 + i\omega) \right] \overset{+}{A}_1(\eta;\omega)\, \overset{+}{n}_1(\eta) - \exp\left[\frac{d}{\eta}(1 + i\omega) \right] \overset{+}{A}_1(\omega;-\eta)\, \overset{+}{n}_1(-\eta) + \\ & + \overset{+}{A}_2(\omega;-\eta)\, \overset{+}{n}_2(-\eta) - \overset{+}{A}_2(\eta;\omega)\, \overset{+}{n}_2(\eta)\,. \end{aligned}$$

Here and below, unless stipulated otherwise, we assume that $\mu \in \mathbf{R}$.

Taking into account Eq. (11), we rearrange system (16) to a system of singular integral equations with a Cauchy kernel:

$$\frac{1}{\sqrt{\pi}} \varphi(\mu;\omega) + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \eta n(\eta;\omega) \frac{d\eta}{\eta - \mu} + \exp(\mu^2)\, \Lambda(\mu;\omega)\, n(\mu;\omega) = 2U\mu\,,$$

$$\frac{1}{\sqrt{\pi}}\psi(\mu;\omega)+\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\eta m(\eta;\omega)\frac{d\eta}{\eta-\mu}+\exp(\mu^2)\Lambda(\mu;\omega)m(\mu;\omega)=2U_d\mu\exp(i\varphi_0).$$

Then we introduce the auxiliary functions:

$$N(z;\omega)=\int_{-\infty}^{\infty}\eta n(\eta;\omega)\frac{d\eta}{n-z},\quad M(z;\omega)=\int_{-\infty}^{\infty}\eta m(\eta;\omega)\frac{d\eta}{\eta-z},\tag{18}$$

which are piecewise analytical functions in the complex plane with a section along the real axis. Using the Sokhotskii formulas [15] for the values of the functions $N(z;\omega)$, $M(z;\omega)$, and $\Lambda(z;\omega)$ on the section, we have:

$$\begin{aligned}\Lambda^+(\mu;\omega)-\Lambda^-(\mu;\omega)&=2\sqrt{\pi}\,i\mu\exp(-\mu^2);\\ N^+(\mu;\omega)-N^-(\mu;\omega)&=2\pi i\mu n(\mu;\omega);\\ M^+(\mu;\omega)-M^-(\mu;\omega)&=2\pi i\mu m(\mu;\omega).\end{aligned}\tag{19}$$

By means of Eqs. (18) and (19) we pass from the system of integral equations to a system of Riemann scalar boundary-value problems:

$$\Lambda^+(\mu;\omega)[\varphi(\mu;\omega)+N^+(\mu;\omega)-2\sqrt{\pi}U\mu]=$$

$$=\Lambda^-(\mu;\omega)[\varphi(\mu;\omega)+N^-(\mu;\omega)-2\sqrt{\pi}U\mu],$$

$$\Lambda^*(\mu;\omega)[\psi(\mu;\omega)+M^+(\mu;\omega)-2\sqrt{\pi}U_d\mu\exp(i\varphi_0)]=$$

$$=\Lambda^-(\mu;\omega)[\psi(\mu;\omega)+M^-(\mu;\omega)-2\sqrt{\pi}U_d\mu\exp(i\varphi_0)].$$

Let

$$P(z;\omega)=\Lambda(z;\omega)[\varphi(z;\omega)+N(z;\omega)-2\sqrt{\pi}Uz],$$

$$Q(z;\omega)=\Lambda(z;\omega)[\psi(z;\omega)+M(z;\omega)-2\sqrt{\pi}U_d z\exp(i\varphi_0)].$$

According to the theorem of analytic continuation [15], the function $P(z;\omega)$ is analytic in the complex plane, except for a point at infinity at which it has a pole of the first order. Then, following Liouville's theorem [15], the function $P(z;\omega)$ is the polynomial of the first order (c_0+c_1z). Considering that $P(0;\omega)=0$, we obtain $P(z;\omega)=c_1z$, therefore:

$$N(z;\omega)=2\sqrt{\pi}Uz+\frac{c_1z}{\Lambda(z;\omega)}+$$

$$+\frac{2\eta_0z}{z^2-\eta_0^2}\left\{a_1(\eta_0;\omega)n_1(\eta_0)-a_2(\eta_0;\omega)\exp\left[-\frac{d}{\eta_0}(1+i\omega)\right]n_2(\eta_0)\right\}.$$

Similar reasoning yields

$$M(\mu;\omega)=2\sqrt{\pi}Uz\exp(i\varphi_0)+\frac{c_2z}{\Lambda(z;\omega)}+$$

$$+\frac{2\eta_0z}{z^2-\eta_0^2}\left\{\exp\left[-\frac{d}{\eta_0}(1+i\omega)\right]a_1(\eta_0;\omega)n_1(\eta_0)-a_2(\eta_0;\omega)n_2(\eta_0)\right\}.$$

Here c_1 and c_2 are unknown coefficients. It is evident that the solutions obtained have poles of the first order at finite points $\pm\eta_0$ and, in addition, have a pole of the first order at a point at infinity. However, the auxiliary functions $M(z; \omega)$ and $N(z, \omega)$ prescribed by formulas (18) are piecewise analytical functions everywhere in the complex plane with a section along the real axis. Therefore, in order to take the solutions obtained as auxiliary functions, it is necessary and sufficient to eliminate in them singularities revealed previously.

Taking into account the behavior of the function $\Lambda(z; \omega)$ at infinity, we eliminate the pole at a point at infinity in the functions $M(z; \omega)$ and $N(z; \omega)$ and assume that $c_1 = -2\sqrt{\pi}U\omega i$, $c_2 = -2\sqrt{\pi}U_d wi \exp(i\varphi_0)$. It is easy to see that now $M(z; \omega) \sim 1/z$ and $N(z; \omega) \sim 1/z$. It remains to eliminate the poles of the first order at finite points $\pm\eta_0$ in the functions $M(z; \omega)$ and $N(z; \omega)$. In order to eliminate the poles with the properties of the functions $M(z; \omega)$, $N(z; \omega)$, and $\Lambda(z; \omega)$ taken into account, it is necessary and sufficient to eliminate the pole at point η_0, and then the pole at point $-\eta_0$ is eliminated automatically. Having expanded the function $\Lambda(z; \omega)$ into series in the vicinity of point η_0 (we note that the function $\Lambda(z; \omega)$ is analytic at point η_0 and therefore we obtain a Taylor series) and allowing for the behavior of $M(z; \omega)$ in the vicinity of this point, we write a system of two linear equations:

$$a_1(\eta_0; \omega)\, n_1(\eta_0) - \exp\left[-\frac{d}{\eta_0}(1 + i\omega)\right] a_2(\eta_0; \omega)\, n_2(\eta_0) = -\frac{c_1}{\lambda'_{\text{cig.f}}(\eta_0)},$$

$$\exp\left[-\frac{d}{\eta_0}(1 + i\omega)\right] a_1(\eta_0; \omega)\, n_1(\eta_0) - a_2(\eta_0; \omega)\, n_2(\eta_0) = -\frac{c_2}{\lambda'_{\text{cig.f}}(\eta_0)}.$$

Solution of this system gives the coefficients for the discrete spectrum:

$$a_1(\eta_0; \omega) = \frac{c_2 \exp[-d(1 + i\omega)/\eta_0] - c_1}{\lambda'_c(\eta_0)\, n_1(\eta_0)\,(1 - \exp[-2d(1 + i\omega)/\eta_0])};$$
$$a_2(\eta_0; \omega) = \frac{c_2 - c_1 \exp[-d(1 + i\omega)/\eta_0]}{\lambda'_c(\eta_0)\, n_2(\eta_0)\,(1 - \exp[-2d(1 + i\omega)/\eta_0])}. \tag{20}$$

Now we find the coefficients for the continuous spectrum. To do this, we use the Sokhotskii formulas (18) and obtain a system of linear equations:

$$n(\mu; \omega) = -\frac{c_1 \mu \exp(-\mu^2)}{\sqrt{\pi}\,\Lambda^+(\mu; \omega)\,\Lambda^-(\mu; \omega)}, \quad m(\mu; \omega) = -\frac{c_2 \mu \exp(-\mu^2)}{\sqrt{\pi}\,\Lambda^+(\mu; \omega)\,\Lambda^-(\mu; \omega)}.$$

Here the functions $n(\mu; \omega)$ and $m(\mu; \omega)$ are prescribed by relations (17).

Having solved this system, we write expressions for the coefficients of the continuous spectrum:

$$A_1^+(\eta; \omega) = \frac{(c_1 - c_2)\,\eta \exp(-\eta^2)}{\sqrt{\pi}\,\Lambda^+(\eta; \omega)\,\Lambda^-(\eta; \omega)\, n_1^+(\eta)\,(1 - \exp[-2d(1 + i\omega)/\eta])},$$
$$A_2^+(\eta; \omega) = \frac{(c_1 \exp[-d(1 + i\omega)/\eta] - c_2)\,\eta \exp(-\eta^2)}{\sqrt{\pi}\,\Lambda^+(\eta; \omega)\,\Lambda^-(\eta; \omega)\, n_2^+(\eta)\,(1 - \exp[-2d(1 + i\omega)/\eta])}. \tag{21}$$

Thus, all the coefficients of expansion are found in explicit form and are prescribed by formulas (21) and (22). The fact that the expansion is a solution of the initial boundary-value problem is verified immediately. The uniqueness of this solution follows from the impossibility of nontrivial expansion of zero in the eigenvectors of the characteristic equation. Consequently, the solution of the initial boundary-value problem in the form of expansion (13) is established.

4. Analysis of Results. In order to analyze the results obtained, we consider the following limiting cases.

1. Let $\omega \ll 1$. We expand the dispersion function $\Lambda(z; \omega)$ into a Laurent series in the vicinity of a point at infinity:

$$\Lambda(z; \omega) = -\frac{1}{2z^2} + i\omega + o\left(\frac{1}{z^2}\right), \quad |z| \to \infty. \tag{22}$$

From formula (22) we find the eigenvalues of the discrete spectrum:

$$\eta_0 = \pm \frac{i - 1}{2\sqrt{\omega}}.$$

Thus, at small ω the eigenvalues of the discrete spectrum tend to infinity.

2. Let $|\eta_0| \ll d$, $\omega \ll 1$, $1 \ll x$. In this case (outside the Knudsen layer) only a discrete mode remains. By virtue of rapid exponential decay, the continuous mode disappears. Expansion (13) has the form

$$\Psi(x, \mu, \omega) = a_1(\eta_0; \omega)\, \Phi_1(\eta_0, \mu) \exp\left[-\frac{x}{\eta_0}(1 + i\omega)\right].$$

Suppose that

$$\delta_n = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\mu^2)\, a_1(\eta_0; \omega)\, \Phi_1(\eta_0, \mu) \exp\left[-\frac{x}{\eta_0}(1 + i\omega)\right] d\mu.$$

With allowance for the boundary conditions we obtain

$$\delta_n = -\frac{2iU\omega \exp\left[-\frac{x}{\eta_0}(1 + i\omega)\right]}{\lambda_c'(\eta_0)} [\lambda_c'(\eta_0) - 1].$$

Taking into account the behavior of the function $\Lambda(z; \omega)$ at small ω, we write an expression for the relative concentration

$$\mathrm{Re}\,\delta_n = -\frac{U \exp[-x\sqrt{\omega}(1 - \omega)]}{2\sqrt{\omega}} [(1 - \omega)\sin(x\sqrt{\omega}(1 + \omega)) - (1 + \omega)\cos(x\sqrt{\omega}(1 + \omega))].$$

3. Let $1 \ll d \le |\eta_0|$, $\omega \ll 1$, $1 \ll x$, $(d - x) \gg 1$. Expansion (13) takes the form

$$\Psi(x, \mu, \omega) = a_1(\eta_0; \omega) \left\{\Phi_1(\eta_0; \mu) \exp\left[-\frac{x}{\eta_0}(1 + i\omega)\right] + \right.$$
$$\left. + \alpha \Phi_2(\eta_0; \mu) \exp\left[-\frac{d - x}{\eta_0}(1 + i\omega)\right]\right\},$$

where

$$\alpha = \frac{a_2(\eta_0; \omega)}{a_1(\eta_0; \omega)} = \frac{n_1(\eta_0)}{n_2(\eta_0)} \exp\left[-\frac{d(1 + i\omega)}{\eta_0}\right].$$

It is evident that the coefficient α characterizes the reflected wave. In this case δ_n is of the form

$$\delta_n = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\mu^2)\, a_1(\eta_0; \omega)\, \Phi_1(\eta_0, \mu) \exp\left[-\frac{x}{\eta_0}(1 + i\omega)\right] d\mu +$$

$$+ \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\mu^2)\, a_2(\eta_0; \omega)\, \Phi_2(\eta_0, \mu) \exp\left[- \frac{d - x}{\eta_0} (1 + i\omega) \right] d\mu .$$

Consequently:

$$\delta_n = \frac{2iU\omega}{\lambda'_{\mathrm{eig.f}}(\eta_0)} [\lambda_{\mathrm{eig.f}}(\eta_0) - 1] \frac{\exp\left[- \frac{x}{\eta_0} (1 + i\omega) \right] + \exp\left[- \frac{2d - x}{\eta_0} (1 + i\omega) \right]}{\exp\left[- \frac{2d}{\eta_0} (1 + i\omega) \right] - 1} .$$

Therefore, the expression for the relative concentration will take the following form:

$$\mathrm{Re}\, \delta_n = \frac{r_1 r_2}{(r_1 a_2 - 1)^2 + (r_2 b_2)^2} \left\{ a_1 + r_2 [(a_1 a_2 + b_1 b_2)(a_2 - 1) + (b_2 a_1 - a_2 b_1) b_2] \right\} .$$

Here

$$a_1 = \cos(x\sqrt{\omega}\, [1 + \omega]) ; \quad b_1 = \sin(x\sqrt{\omega}\, [1 + \omega]) ; \quad r_1 = \exp[-x\sqrt{\omega}\,(1 - \omega)] ;$$

$$a_2 = \cos(2d\sqrt{\omega}\, [1 + \omega]) ; \quad b_2 = \sin(2d\sqrt{\omega}\, [1 + \omega]) ; \quad r_2 = \exp[2d\sqrt{\omega}\,(1 - \omega)] .$$

Thus, in the present work a method is developed that allows one to obtain accurate solutions of boundary-value problems in a layer for nonstationary model kinetic equations with mirror boundary conditions.

The separation of variables leads to a characteristic system. The eigenfunctions for the discrete and continuous spectra of the characteristic system are found. An expansion of the solution of the initial boundary-value problem in eigenfunctions is established. The unknown coefficients of the expansion are obtained in explicit form.

Problems with boundary conditions (4) can be used in solving the most diverse problems of the kinetic theory of gas and plasma, in the theory of neutron and electron transfer, in theoretical astrophysics, etc.

The present work was carried out with partial financial support from the Russian Fund for Fundamental Scientific Research (project code 99-01-00336).

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